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Robust stochastic control and equivalent martingale measures

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Abstract

We study a class of robust, or worst case scenario, optimal control problems for jump diffusions. The scenario is represented by a probability measure equivalent to the initial probability law. We show that if there exists a control that annihilates the noise coefficients in the state equation and a scenario which is an equivalent martingale measure for a specific process which is related to the control-derivative of the state process, then this control and this probability measure are optimal. We apply the result to the problem of consumption and portfolio optimization under model uncertainty in a financial market, where the price process $S(t)$ of the risky asset is modeled as a geometric Itô-Lévy process. In this case the optimal scenario is an equivalent local martingale measure of $S(t)$. We solve this problem explicitly in the case of logarithmic utility functions.

1 Introduction

During the last decade there has been an increasing awareness of the importance of taking *model uncertainty* into account when dealing with mathematical models. See e.g. [HS] and the references therein. A general feature of model uncertainty is the recognition of the uncertainty about the underlying probability law, or scenario, for the model. This leads to the study of *robust* models, where one seeks an optimal strategy among a family of possible scenarios. A special case is the problem of optimal control in the worst possible scenario. This is the topic of this paper. We consider a class of scenarios which is basically the set of

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all probability measures Q which are absolutely continuous with respect to a given reference measure P, and we study an optimal control problem for a jump diffusion under the worst possible scenario.

Mathematically this leads to a *stochastic differential game* between the controller and the "environment" who chooses the scenario. Assuming the system is Markovian and using the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, we show that if there exists a control $\hat{u}$ which annihilates the noise coefficients of the system and a scenario $\hat{Q}$ which is an equivalent martingale measure for a specific process which is related to the control-derivative of the state process, then this control is optimal for the controller and the scenario $\hat{Q}$ is a worst case scenario.

We then apply this result to the problem of optimal consumption and portfolio under model uncertainty in a financial market, where the price process $S(t)$ of the risky asset is modeled as a geometric Itô-Lévy process. In this case the optimal scenario is an equivalent local martingale measure of $S(t)$. We solve this problem explicitly in the case of logarithmic utility functions.

Robust control problems and worst case scenario problems have been studied by many researchers. We mention in particular the paper [BMS], where the following approach is used: The authors first fix a strategy and prove the existence of a corresponding optimal scenario Q^*, and then subsequently use BSDEs to study the optimal strategy problem for a fixed scenario. Robust control problems for possibly non-Markovian systems can also be studied by means of stochastic maximum principles (see [AØ]).

2 Robust optimal control

2.1 Stochastic differential game approach

Consider a controlled jump diffusion $X(t) = X^u(t)$ in $\mathbb{R}$ of the form

$$\begin{aligned} dX(t) &= b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}_0} \gamma(t, X(t), u(t), z)\tilde{N}(dt, dz) \; ; \; X(0) = x \in \mathbb{R} \; ; \; t \in [0, T] \end{aligned} \tag{2.1}$$

where $T > 0$ is a fixed constant. Here $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$ is the compensated Poisson random measure of a Lévy process with jump measure $N(\cdot, \cdot)$ and Lévy measure $\nu(\cdot)$, and $B(t)$ is an independent Brownian motion. Both $\tilde{N}(\cdot, \cdot)$ and $B(\cdot)$ live on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$.

The process $u(t) = (u_1(t), \cdot \cdot \cdot, u_m(t)) \in \mathbb{R}^m$ is the control process of the agent and $b(s, x, u)$, $\sigma(s, x, u)$ and $\gamma(s, x, u, z)$; $s \in [0, T]$, $x \in \mathbb{R}$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, are assumed to be $\mathcal{C}^1$ functions with respect to u.

The *scenario* of $X(t)$ is determined by a (positive) measure Q^θ of the form

$$dQ^\theta(\omega) = K^\theta(T)dP(\omega) \text{ on } \mathcal{F}_T, \tag{2.2}$$

where

$$dK^\theta(t) = -K^\theta(t^-)\left[\theta_0(t)dB(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\tilde{N}(dt,dz)\right] \; ; \; t \in [0,T]$$
$$K^\theta(0) = k > 0. \tag{2.3}$$

Here $\theta = \theta(t,z) = (\theta_0(t), \theta_1(t,z)) \in \mathbb{R}^2$ is the *scenario control*, assumed to be $\mathcal{F}_t$-predictable and such that

$$E[K^\theta(T)] = K^\theta(0) =: k > 0. \tag{2.4}$$

Let V, Θ be two sets such that $u(t) \in V$ and $\theta(t,z) \in \Theta$ for all t, z and let $\mathcal{U}$, $\mathcal{A}$ be given families of admissible u-controls and θ-controls, respectively.

Define the process $Y(t) = Y^{\theta,u}(t) := (K^\theta(t), X^u(t))$. Then $Y(t)$ is a controlled jump diffusion with generator

$$\begin{aligned}
A^{\theta,u}\varphi(t,y) &= A^{\theta,u}\varphi(t,k,x) \\
&= b(t,x,u)\frac{\partial\varphi}{\partial x} + \frac{1}{2}k^2\theta_0^2\frac{\partial^2\varphi}{\partial k^2} + \frac{1}{2}\sigma^2(t,x,u)\frac{\partial^2\varphi}{\partial x^2} \\
&- \theta_0 k\sigma(t,x,u)\frac{\partial^2\varphi}{\partial k\partial x} \\
&+ \int_{\mathbb{R}_0} \{\varphi(t, k - k\theta_1(z), x + \gamma(t,x,u,z)) - \varphi(t,k,x) \\
&+ k\theta_1(z)\frac{\partial\varphi}{\partial k}(t,k,x) - \gamma(t,x,u,z)\frac{\partial\varphi}{\partial x}(t,k,x)\}\nu(dz) \; ; \; \varphi \in \mathcal{C}^{1,2,2}(\mathbb{R}^3).
\end{aligned} \tag{2.5}$$

We refer to [ØS1] for more information about stochastic control of jump diffusions.

Let $f : \mathbb{R}^2 \times V \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that

$$E_{Q^\theta}\left[\int_0^T |f(t,X(t),u(t))|dt + |g(X(T))|\right] < \infty$$

for all $u \in \mathcal{U}$, $\theta \in \mathcal{A}$. Define the *performance functional* by

$$\begin{aligned}
J^{\theta,u}(t,y) &= E_{Q^\theta}^{t,y}\left[\int_t^T f(s,X(s),u(s))ds + g(X(T))\right] \\
&= E^{t,y}\left[\int_t^T K^\theta(s)f(s,X(s),u(s))ds + K^\theta(T)g(X(T))\right],
\end{aligned} \tag{2.6}$$

where $E_{Q^\theta}^{t,y}$ and $E^{t,y}$ denotes expectation with respect to Q^θ and P, respectively, given $Y(t) = y$.

We consider the following robust, or worst case scenario, stochastic control problem

Problem 2.1 *Find $\theta^* \in \mathcal{A}$, $u^* \in \mathcal{U}$ and $\Phi(t,y)$ such that*

$$\Phi(t,y) = \inf_{\theta\in\mathcal{A}}\left(\sup_{u\in\mathcal{U}} J^{\theta,u}(t,y)\right) = J^{\theta^*,u^*}(t,y). \tag{2.7}$$

2.2 The main theorem

We now formulate our main result:

Theorem 2.2 *Suppose there exist a $\mathcal{C}^1$ function $\psi(t,x)$ and feedback controls $\hat{u}=\hat{u}(t,x)\in\mathcal{U}$, $\hat{\theta}=(\hat{\theta}_0(t,x),\hat{\theta}_1(t,x,z))\in\mathcal{A}$ such that*

$$\sigma(t,x,\hat{u}(t,x))=\gamma(t,x,\hat{u}(t,x),z)=0 \text{ for all } t,x,z \tag{2.8}$$

and

$$\left[\hat{\theta}_0(t,x)\frac{\partial\sigma}{\partial u_i}(t,x,\hat{u}(t,x))+\int_{\mathbb{R}_0}\hat{\theta}_1(t,x,z)\frac{\partial\gamma}{\partial u_i}(t,x,\hat{u}(t,x),z)\nu(dz)-\frac{\partial b}{\partial u_i}(t,x,\hat{u}(t,x))\right]\frac{\partial\psi}{\partial x}(t,x)$$
$$=\frac{\partial f}{\partial u_i}(t,x,\hat{u}(t,x))\ ;\quad \text{for all } t,x,\quad i=1,2,\dots,m. \tag{2.9}$$

Then $(\hat{u},\hat{\theta})$ is an optimal pair for the robust control problem (2.7) and the value function is given by $\Phi(t,k,x)=k\psi(t,x)$; provided that $\psi(t,x)$ is the solution of the PDE

$$\frac{\partial\psi}{\partial t}(t,x)+b(t,x,\hat{u}(t,x))\frac{\partial\psi}{\partial x}(t,x)+f(t,x,\hat{u}(t,x))=0\ ;\ (t,x)\in[0,T]\times\mathbb{R} \tag{2.10}$$

$$\psi(T,x)=g(x)\ ;\ x\in\mathbb{R}. \tag{2.11}$$

Proof. We apply Theorem 3.2 in [MØ]. Maximizing $u\to A^{\theta,u}\varphi(t,k,x)+kf(t,x,u)$ with respect to u gives the following first order conditions for an optimal $\hat{u}$:

$$\begin{aligned}&\frac{\partial b}{\partial u_i}(t,x,\hat{u})\frac{\partial\varphi}{\partial x}+\sigma(t,x,\hat{u})\frac{\partial\sigma}{\partial u_i}(t,x,\hat{u})\frac{\partial^2\varphi}{\partial x^2}-\theta_0k\frac{\partial\sigma}{\partial u_i}(t,x,\hat{u})\frac{\partial^2\varphi}{\partial k\partial x}\\&\quad+\int_{\mathbb{R}_0}\left\{\frac{\partial\varphi}{\partial x}(t,k-k\theta_1(z),x+\gamma(t,x,\hat{u},z))-\frac{\partial\varphi}{\partial x}(t,k,x)\right\}\frac{\partial\gamma}{\partial u_i}(t,x,\hat{u},z)\nu(dz)\\&\quad+k\frac{\partial f}{\partial u_i}(t,x,\hat{u})=0;\quad i=1,2,\dots,m.\end{aligned} \tag{2.12}$$

Minimizing $A^{\theta,\hat{u}}\varphi(t,k,x)+kf(t,x,\hat{u})$ with respect to $\theta=(\theta_0,\theta_1(z))$, we get the following first order conditions for optimal $\hat{\theta}_0,\hat{\theta}_1(z)$:

$$k^2\hat{\theta}_0\frac{\partial^2\varphi}{\partial k^2}-k\sigma(t,x,\hat{u})\frac{\partial^2\varphi}{\partial k\partial x}=0 \tag{2.13}$$

and

$$\int_{\mathbb{R}_0}\left\{\frac{\partial\varphi}{\partial k}(t,k-k\hat{\theta}_1(z),x+\gamma(t,x,\hat{u},z))-\frac{\partial\varphi}{\partial k}(t,k,x)\right\}\nu(dz)=0. \tag{2.14}$$

Let us try a value function of the form

$$\varphi(t,k,x)=k\psi(t,x). \tag{2.15}$$

Then (2.12)-(2.14) get the form

$$\begin{aligned}
&\frac{\partial b}{\partial u_i}(t,x,\hat{u})\frac{\partial \psi}{\partial x} + \sigma(t,x,\hat{u})\frac{\partial \sigma}{\partial u_i}(t,x,\hat{u})\frac{\partial^2 \psi}{\partial x^2} - \hat{\theta}_0 \frac{\partial \sigma}{\partial u_i}(t,x,\hat{u})\frac{\partial \psi}{\partial x} \\
&\quad + \int_{\mathbb{R}_0} \left\{ (1-\hat{\theta}_1(z))\frac{\partial \psi}{\partial x}(t, x+\gamma(t,x,\hat{u},z)) - \frac{\partial \psi}{\partial x}(t,x) \right\} \frac{\partial \gamma}{\partial u_i}(t,x,\hat{u},z)\nu(dz) \\
&\quad + \frac{\partial f}{\partial u_i}(t,x,\hat{u}) = 0; \quad i = 1,2,\dots,m,
\end{aligned} \tag{2.16}$$

$$\sigma(t,x,\hat{u})\frac{\partial \psi}{\partial x}(t,x) = 0, \tag{2.17}$$

and

$$\int_{\mathbb{R}_0} \{\psi(t, x+\gamma(t,x,\hat{u},z)) - \psi(t,x)\}\nu(dz) = 0. \tag{2.18}$$

Suppose there exists a Markov control $\hat{u} = \hat{u}(t,x)$ such that

$$\sigma(t,x,\hat{u}(t,x)) = \gamma(t,x,\hat{u}(t,x),z) = 0 \text{ for all } z \in \mathbb{R}_0. \tag{2.19}$$

Then (2.17)-(2.18) are satisfied, and (2.16) gets the form

$$\begin{aligned}
&\left[\hat{\theta}_0(t,x)\frac{\partial \sigma}{\partial u_i}(t,x,\hat{u}) + \int_{\mathbb{R}_0} \hat{\theta}_1(t,x,z)\frac{\partial \gamma}{\partial u_i}(t,x,\hat{u},z)\nu(dz) - \frac{\partial b}{\partial u_i}(t,x,\hat{u})\right] \frac{\partial \psi}{\partial x}(t,x) \\
&\quad = \frac{\partial f}{\partial u_i}(t,x,\hat{u}); \quad i = 1,2,\dots,m.
\end{aligned} \tag{2.20}$$

Suppose $\hat{u}, \hat{\theta}$ satisfy (2.19)-(2.20). Then by Theorem 3.2 in [MØ] we are required to have

$$\frac{\partial \varphi}{\partial t}(t,y) + A^{\hat{\theta},\hat{u}}\varphi(t,y) + kf(t,x,\hat{u}) = 0 \ ; \ t < T.$$

By (2.5) and (2.19), this gives the equation

$$\frac{\partial \psi}{\partial t}(t,x) + b(t,x,\hat{u}(t,x))\frac{\partial \psi}{\partial x}(t,x) + f(t,x,\hat{u}(t,x)) = 0 \ ; \ t < T, \tag{2.21}$$

with terminal condition

$$\psi(T,x) = g(x) \ ; \ x \in \mathbb{R}. \tag{2.22}$$

This completes the proof. □

Remark 2.3 Note that $\hat{u}(t,x)$ and $\hat{\theta}(t,x)$ might depend on $\frac{\partial \psi}{\partial x}(t,x)$. Hence equation (2.10) *is in general a nonlinear PDE in the unknown function ψ.*

2.3 Equivalent local martingale measures

The following definition is motivated by applications in mathematical finance:

Definition 2.4 *Let $S(t)$ be an Itô-Lévy process of the form*

$$dS(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}_0} \lambda(t,z)\tilde{N}(dt,dz)$$

for predictable processes $\alpha(t)$, $\beta(t)$, $\lambda(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}_0$.

A probability measure Q on $\mathcal{F}_T$ is called an equivalent local martingale measure (ELMM) for $S(\cdot)$ if $Q \sim P$ (i.e. $Q \ll P$ and $P \ll Q$) and $\{S(t)\}_{t\in[0,T]}$ is a local martingale with respect to Q.

It is well-known (see e.g. [ØS1, Theorem 1.31]) that a measure Q^θ of the form (2.2)-(2.4) with $k = 1$ is an ELMM for $S(\cdot)$ if and only if

$$\theta_0(t)\beta(t) + \int_{\mathbb{R}_0} \theta_1(t,z)\lambda(t,z)\nu(dz) = \alpha(t) \; ; \; t \in [0,T]. \tag{2.23}$$

Suppose that $\frac{\partial\psi}{\partial x} \neq 0$. Then, in view of Definition 2.4, the measure $Q_{\hat{\theta}}$ of the form (2.2)-(2.4) with $k = 1$, where $\hat{\theta}$ is a scenario control satisfying (2.9), is a ELMM for all the processes $G_i(t)$, given by

$$\begin{aligned} dG_i(t) := &\left[\frac{\partial b}{\partial u_i}(t,\hat{X}(t),\hat{u}(t)) + \left(\frac{\partial\psi}{\partial x}(t,\hat{X}(t))\right)^{-1} \frac{\partial f}{\partial u_i}(t,\hat{X}(t),\hat{u}(t))\right] dt \\ &+ \frac{\partial\sigma}{\partial u_i}(t,\hat{X}(t),\hat{u}(t))dB(t) + \int_{\mathbb{R}_0} \frac{\partial\gamma}{\partial u_i}(t,\hat{X}(t),\hat{u}(t),z)\tilde{N}(dt,dz); \quad i = 1,2,\dots,m. \end{aligned} \tag{2.24}$$

where $\hat{u}(t) = \hat{u}(t,\hat{X}(t))$ and $\hat{X}(t) = X^{\hat{u}}(t)$; $t \in [0,T]$.

3 Example

Suppose we have a financial market with a risk free asset with unit price $S_0(t) = 1$ and a risky asset with unit price $S(t)$ given by

$$\begin{aligned} &dS(t) = S(t^-)\left[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t,z)\tilde{N}(dt,dz)\right] \; ; \; t \in [0,T] \\ &S(0) > 0, \end{aligned} \tag{3.1}$$

where $b_0(t)$, $\sigma_0(t)$ and $\gamma_0(t,z)$ are bounded deterministic functions, $\gamma_0(t,z) > -1$. If we apply a *portfolio* $\pi(t)$, representing the proportion of the total wealth $X(t)$ invested in the risky asset at time t and a *relative consumption rate* $\lambda(t) \geq 0$, the corresponding wealth process $X(t) = X^{\lambda,\pi}(t)$ will have the dynamics

$$\begin{aligned} dX(t) &= \pi(t)X(t^-)\left[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t,z)\tilde{N}(dt,dz)\right] - \lambda(t)X(t)dt\ ;\ t \in [0,T], \\ X(0) &= x > 0. \end{aligned} \tag{3.2}$$

We say that the pair $u = (\lambda, \pi)$ is an *admissible control* if λ and π are $\mathcal{F}$-predictable, $\lambda \geq 0$, $\pi(t)\gamma_0(t,z) > -1$ and $\int_0^T (\pi^2(t) + \lambda(t) + \int_{\mathbb{R}_0} |\log(1 + \pi(t)\gamma_0(t,z))|\nu(dz))dt < \infty$ a .s. Note that under these conditions the unique solution $X(t)$ of (3.2) is given by

$$\begin{aligned} X(t) =& x\exp(\int_0^t \{\pi(s)b_0(s) - \lambda(s) - \frac{1}{2}\pi^2(s)\sigma_0^2(s)\}ds + \int_0^t \pi(s)\sigma_0(s)dB_s \\ &+ \int_0^t \int_{\mathbb{R}_0} (\log(1 + \pi(s)\gamma_0(s,z)) - \pi(s)\gamma_0(s,z))\nu(dz)ds \\ &+ \int_0^t \int_{\mathbb{R}_0} (\log(1 + \pi(s)\gamma_0(s,z))\tilde{N}(ds,dz)); t \in [0,T] \end{aligned}$$

In particular, $X(t) > 0$ for all $t \in [0,T]$.

Let $U_1 : [0,T] \times [0,\infty) \to \mathbb{R}$, $U_2 : [0,\infty) \to \mathbb{R}$ be two given $\mathcal{C}^1$ functions. We assume that $c \to U_1(t,c)$ and $x \to U_2(x)$ are strictly increasing, concave functions (utility functions) for all $t \in [0,T]$. We also assume that $c \mapsto \frac{\partial U_1}{\partial c}(t,c)$ is strictly decreasing and that $\lim_{c\to+\infty} \frac{\partial U_1}{\partial c}(t,c) = 0$ for all $t \in [0,T]$. Put $x_0 = \frac{\partial U_1}{\partial c}(t,0)$ and define

$$I(t,x) = \begin{cases} 0 & \text{for } x \geq x_0 \\ \left(\frac{\partial U_1}{\partial c}(t,\cdot)\right)^{-1}(x) & \text{for } 0 \leq x < x_0 \end{cases} \tag{3.3}$$

Define the *absolute consumption rate* at time t by

$$c(t) = \lambda(t)X(t)\ ;\ t \in [0,T]. \tag{3.4}$$

Suppose the performance functional is given by

$$\begin{aligned} J^{\theta,\lambda,\pi}(t,y) &= E_{Q^\theta}^{t,y}\left[\int_t^T U_1(s,c(s))ds + U_2(X(T))\right] \\ &= E^{t,y}\left[\int_t^T K^\theta(s)U_1(s,c(s))ds + K^\theta(T)U_2(X(T))\right]. \end{aligned} \tag{3.5}$$

To solve the problem

$$\Phi(t,y) = \inf_{\theta\in\mathcal{A}}\left(\sup_{(\lambda,\pi)\in\mathcal{U}} J^{\theta,\lambda,\pi}(t,y)\right) = J^{\theta^*,\lambda^*,\pi^*}(t,y) \tag{3.6}$$

we apply Theorem 2.2. Thus we search for a solution $\hat{c}$, $\hat{\pi}$, $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ and $\psi(t,x)$ such that (2.8) and (2.9) hold, when $b(t,x,u) = \pi x b_0(t) - \lambda x$, $\sigma(t,x,u) = \pi x \sigma_0(t)$, $\gamma(t,x,u,z) = \pi x \gamma_0(t,x,z)$ and $f(t,x,u) = U_1(t,c)$, $g(x) = U_2(x)$, $u = (\lambda, \pi)$, $c = \lambda x$.

We see that (2.8) holds with $\hat{\pi} = 0$, for all $\hat{c}$. Writing $\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) = \left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \pi}\right)$, equation (2.9) becomes

$$\frac{\partial \psi}{\partial x}(t,x) = \frac{\partial U_1}{\partial c}(t, \hat{c}(t,x)) \tag{3.7}$$

and

$$\hat{\theta}_0(t,x)\sigma_0(t) + \int_{\mathbb{R}_0} \theta_1(t,x,z)\gamma_0(t,z)\nu(dz) = b_0(t). \tag{3.8}$$

The equation (2.10)-(2.11) for ψ gets the form

$$\begin{aligned} &\frac{\partial \psi}{\partial t}(t,x) - \hat{c}(t,x)\frac{\partial U_1}{\partial c}(t, \hat{c}(t,x)) + U_1(t, \hat{c}(t,x)) = 0 \; ; \; t < T \\ &\psi(T,x) = U_2(x) \end{aligned} \tag{3.9}$$

which has the solution

$$\psi(t,x) = U_2(x) + \int_t^T \{U_1(s, \hat{c}(s,x)) - \hat{c}(s,x)\frac{\partial U_1}{\partial c}(s, \hat{c}(s,x))\} ds. \tag{3.10}$$

In this case the processes G_i defined in (2.24) are

$$\begin{aligned} dG_1(t) &= \hat{X}(t)\left[-1 + \left(\frac{\partial \psi}{\partial x}(t, \hat{X}(t))\right)^{-1} \frac{\partial U_1}{\partial c}(t, \hat{c}(t, \hat{X}(t)))\right] dt = 0 \\ dG_2(t) &= \hat{X}(t^-)\left[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma_0(t,z)\tilde{N}(dt,dz)\right] \\ &= \frac{\hat{X}(t^-)}{S(t^-)} dS(t). \end{aligned}$$

We conclude that the optimal scenario control is to choose $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ such that (3.8) holds, i.e. such that $Q_{\hat{\theta}}$ is an ELMM for $S(t)$.

The corresponding optimal portfolio is to choose $\hat{\pi} = 0$ (no money in the risky asset). This is intuitively reasonable, because if the price process is a martingale, there is no money to be gained by investing in this asset.

Finally, to find the optimal consumption rate $\hat{c}(t,x)$ we combine (3.7) and (3.10) : From (3.7) and (3.3) we have

$$\hat{c}(t,x) = I\left(t, \frac{\partial \psi}{\partial x}(t,x)\right). \tag{3.11}$$

Differentiating (3.10) we therefore get

$$\frac{\partial \psi}{\partial t}(t,x) = -U_1\left(t, I\left(t, \frac{\partial \psi}{\partial x}(t,x)\right)\right) + I\left(t, \frac{\partial \psi}{\partial x}(t,x)\right)\frac{\partial \psi}{\partial x}(t,x) \; ; \; t < T. \tag{3.12}$$

This is a nonlinear first order partial differential equation in $\psi(t, x)$. Together with the terminal value (obtained from (3.10))

$$\psi(T, x) = U_2(x) \tag{3.13}$$

this determines $\psi(t, x)$ uniquely. To summarize, we have proved

Theorem 3.1 The *optimal (i.e. worst case) scenario control for the problem* (3.6) is to choose $\theta^* = (\hat{\theta}_0, \hat{\theta}_1)$ such that (3.8) holds, which is equivalent to saying that the measure $Q_{\hat{\theta}}$ is an ELMM for the price process $S(t)$ given by (3.1).
The *optimal portfolio* under this scenario is to choose $\hat{\pi} = 0$ (no money in the risky asset).
The *optimal consumption rate* $\hat{c}(t, x)$ under this scenario is given by (3.11), i.e.

$$\frac{\partial U_1}{\partial x}(t, \hat{c}(t, x)) = \frac{\partial \psi}{\partial x}(t, x) \tag{3.14}$$

where $\psi(t, x)$ is the solution of (3.12)-(3.13). The corresponding value function is, by (2.15),

$$\Phi(t, k, x) = k\psi(t, x). \tag{3.15}$$

This is an extension of the result in [ØS2].

A special case. To illustrate the content of Theorem 3.1, we consider the special case when U_1 and U_2 are logarithmic utility functions, i.e.

$$U_1(s, c) = U_1(c) = \ln c; \quad U_2(x) = \lambda \ln x \tag{3.16}$$

where $\lambda > 0$ is constant. Then $U_1'(c) = \frac{1}{c}$ and, by (3.3) $I(x) = \frac{1}{x}$. Therefore equation (3.12) gets the form

$$\frac{\partial \psi}{\partial t}(t, x) = \ln\left(\frac{\partial \psi}{\partial x}(t, x)\right) + 1. \tag{3.17}$$

Set

$$h(t, x) = \psi(t, x) - t.$$

Then

$$\frac{\partial h}{\partial t}(t, x) = \ln\left(\frac{\partial h}{\partial x}(t, x)\right).$$

Set

$$H(t, x) = \frac{\partial h}{\partial x}(t, x).$$

Then H satisfies the nonlinear PDE:

$$H(t, x)\frac{\partial H}{\partial t}(t, x) = \frac{\partial H}{\partial x}(t, x); \; t \leq T \tag{3.18}$$

We assign the terminal condition

$$H(T, x) = \frac{\lambda}{x}$$

and try to solve this equation by setting

$$H(t,x) = H_1(t)\frac{1}{x}.$$

Substituting into (3.18) gives

$$H_1'(t) = -1.$$

Since $H_1(T) = \lambda$, this gives the solution

$$H(t,x) = \frac{T-t+\lambda}{x}$$

and hence

$$h(t,x) = (T-t+\lambda)\ln x + C(t)$$

for some function $C(t)$. This gives

$$\psi(t,x) = (T-t+\lambda)\ln x + t + C(t)$$

and hence

$$\frac{\partial\psi}{\partial t}(t,x) = -\ln x + C'(t) + 1$$

while

$$\ln\left(\frac{\partial\psi}{\partial x}(t,x)\right) = \ln(T-t+\lambda) - \ln x.$$

Using (3.17), we get

$$C'(t) = \ln(T-t+\lambda) - 1$$

or

$$C(t) = -(T-t+\lambda)\ln(T-t+\lambda) + (T-t+\lambda) - t + C_0$$

Hence

$$\psi(t,x) = (T-t+\lambda)[\ln x - \ln(T-t+\lambda) + 1] + C_0.$$

Requiring

$$\psi(T,x) = U_2(x) = \lambda\ln x$$

leads to the condition

$$C_0 = \lambda\ln\lambda - \lambda.$$

Hence the solution is

$$\psi(t,x) = (T-t+\lambda)[\ln x - \ln(T-t+\lambda)] + \lambda\ln\lambda + T - t \tag{3.19}$$

and the optimal consumption rate is then

$$\hat{c}(t,x) = \frac{x}{T-t+\lambda}. \tag{3.20}$$

In particular, we see that the consumption increases with time, which makes sense, because a large early consumption reduces the growth for the whole remaining time period.

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